Note on the Riemann Hypothesis

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Abstract

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In 2011, Solé and and Planat stated that the Riemann Hypothesis is true if and only if the Dedekind inequality $\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(q_n)$ is satisfied for all primes $q_n > 3$, where $\theta(x)$ is the Chebyshev function, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and $\zeta(x)$ is the Riemann zeta function. We can deduce from that paper, if the Riemann Hypothesis is false, then the Dedekind inequality is not satisfied for infinitely many prime numbers q_n. Using this result, we prove the Riemann Hypothesis is true when $(1 - \frac{0.15}{\log^3 x})^{\frac{1}{x}} \times x^{\frac{1}{x}} \geq 1 + \frac{\log(1-\frac{0.15}{\log^3 x}) + \log x}{x}$ is always satisfied for every sufficiently large positive number x. However, we know that inequality is trivially satisfied for every sufficiently large positive number x. In this way, we show the Riemann Hypothesis is true.

Keywords: Riemann Hypothesis, Prime numbers, Dedekind inequality, Chebyshev function, Riemann zeta function
2000 MSC: 11M26, 11A41, 11A25

1. Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x [1]. We denote the n^{th} prime number as q_n. We know the following properties for the Chebyshev function:

Proposition 1.1. *For all $n \geq 2$, we have [2]:*

$$\frac{\theta(q_n)}{\log q_{n+1}} \geq n \times (1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}).$$

Proposition 1.2. *For every $x \geq 19035709163$ [3]:*

$$\theta(x) > (1 - \frac{0.15}{\log^3 x}) \times x.$$

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Besides, we define the prime counting function $\pi(x)$ as

$$\pi(x) = \sum_{p \leq x} 1.$$

We also know this property for the prime counting function:

Proposition 1.3. *For every* $x \geq 19027490297$ *[3]:*

$$\pi(x) > \eta_x$$

where

$$\eta_x = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2 \times x}{\log^3 x} + \frac{5.85 \times x}{\log^4 x} + \frac{23.85 \times x}{\log^5 x} + \frac{119.25 \times x}{\log^6 x} + \frac{715.5 \times x}{\log^7 x} + \frac{5008.5 \times x}{\log^8 x}.$$

In mathematics, $\Psi = n \times \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function, where $q \mid n$ means the prime q divides n. Say $\mathsf{Dedekind}(q_n)$ holds provided

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(q_n).$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, log is the natural logarithm and $\zeta(x)$ is the Riemann zeta function. The importance of this inequality is:

Proposition 1.4. $\mathsf{Dedekind}(q_n)$ *holds for all prime numbers* $q_n > 3$ *if and only if the Riemann Hypothesis is true [4].*

We define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant [5]. We know from the constant H, the following formula:

Proposition 1.5. *We have that [6]:*

$$\sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \frac{1}{q_k}\right) = \gamma - B = H.$$

We know this value of the Riemann zeta function:

Proposition 1.6. *It is known that [6]:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6}.$$

We have the following result:

Proposition 1.7. *For every* $x > -1$ *[7]:*

$$x \geq \log(1 + x).$$

Putting all together yields a proof for the Riemann Hypothesis using the Chebyshev function.

2. Results

Lemma 2.1. *If the Riemann Hypothesis is false, then there are infinitely many prime numbers q_n for which* $\mathsf{Dedekind}(q_n)$ *do not hold.*

Proof. If the Riemann Hypothesis is false, then we consider the function [4]:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \times \log\theta(x) \times \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

We know the Riemann Hypothesis is false, if there exists some natural number $x_0 \geq 5$ such that $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$ [4]. We know the bound [4]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}$$

where f is introduced in the Nicolas paper [1]:

$$f(x) = e^{\gamma} \times \log\theta(x) \times \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

We know when the Riemann Hypothesis is false, then there exists a real number $b < \frac{1}{2}$ and there are infinitely many natural numbers x such that $\log f(x) = \Omega_{+}(x^{-b})$ [1]. According to Hardy and Littlewood, this would mean that $\exists k > 0, \forall y_0, \exists y > y_0 \colon \log f(y) \geq k \times y^{-b}$. The inequality $\log f(y) \geq k \times y^{-b}$ is equivalent to $\log f(y) \geq \left(k \times y^{-b} \times \sqrt{y}\right) \times \frac{1}{\sqrt{y}}$, but we know that $\left(k \times y^{-b} \times \sqrt{y}\right) \geq 1$ is always satisfied starting for some y'_0 such that $y \geq y'_0$. This would be true no matter how small could be the value of k due to $b < \frac{1}{2}$. Hence, if the Riemann Hypothesis is false, then there are infinitely many natural numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{2}{x} = o(\frac{1}{\sqrt{x}})$, then it would be infinitely many natural numbers x_0 such that $\log g(x_0) > 0$ [4]. □

The following is a key Lemma.

Lemma 2.2.

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k})\right) = \log(\zeta(2)) - H.$$

Proof. If we add H to

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k})\right)$$

then we obtain that

$$
\begin{aligned}
H + \sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k}) \right) &= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \frac{1}{q_k} \right) + \sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(\frac{q_k + 1}{q_k}) \right) \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \log(\frac{q_k + 1}{q_k}) \right) \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) + \log(\frac{q_k}{q_k + 1}) \right) \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k^2}{(q_k - 1) \times (q_k + 1)}) \right) \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k^2}{(q_k^2 - 1)}) \right) \\
&= \log(\prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1}) \\
&= \log(\zeta(2))
\end{aligned}
$$

according to the Propositions 1.5 and 1.6. Therefore, the proof is done. □

This is a new criterion based on the Dedekind inequality.

Lemma 2.3. Dedekind(q_n) *holds for all prime numbers* $q_n > 3$ *if and only if the inequality*

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x > q_n\}}(q_k)) \times \log(1 + \frac{1}{q_k}) \right) > B + \log \log \theta(q_n)$$

is satisfied for all prime numbers $q_n > 3$*, where the set* $S = \{x : x > q_n\}$ *contains all the real numbers greater than* q_n *and* χ_S *is the characteristic function of the set* S *(This is the function defined by* $\chi_S(x) = 1$ *when* $x \in S$ *and* $\chi_S(x) = 0$ *otherwise).*

Proof. We start from the inequality:

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q} \right) > \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(q_n).$$

If we apply the logarithm to the both sides of the inequality, then

$$\log(\zeta(2)) + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > \gamma + \log \log \theta(q_n).$$

This is the same as

$$\log(\zeta(2)) - H + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > B + \log \log \theta(q_n)$$

which is

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k}) \right) + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > B + \log \log \theta(q_n)$$

according to the Lemma 2.2. Let's distribute the elements of the inequality to obtain that

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x>q_n\}}(q_k)) \times \log(1+\frac{1}{q_k})\right) > B + \log\log\theta(q_n)$$

when $\mathsf{Dedekind}(q_n)$ holds. The same happens in the reverse implication. □

This is the main insight.

Lemma 2.4. *The Riemann Hypothesis is true if the inequality*

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n.

Proof. The inequality

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x>q_n\}}(q_k)) \times \log(1+\frac{1}{q_k})\right) > B + \log\log\theta(q_n)$$

is satisfied when

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x\geq q_n\}}(q_k)) \times \log(1+\frac{1}{q_k})\right) > B + \log\log\theta(q_n)$$

is also satisfied, where the set $S = \{x : x \geq q_n\}$ contains all the real numbers greater than or equal to q_n. In the inequality

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x\geq q_n\}}(q_k)) \times \log(1+\frac{1}{q_k})\right) > B + \log\log\theta(q_n)$$

only change the value of

$$\log(1+\frac{1}{q_n}) + \log\log\theta(q_n)$$

and

$$\log\log\theta(q_{n+1})$$

between the consecutive primes q_n and q_{n+1}. Hence, it is enough to show that

$$\log(1+\frac{1}{q_n}) + \log\log\theta(q_n) \geq \log\log\theta(q_{n+1})$$

for all sufficiently large prime numbers q_n according to the Lemmas 2.1 and 2.3. Indeed, the inequality

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x\geq q_n\}}(q_k)) \times \log(1+\frac{1}{q_k})\right) > B + \log\log\theta(q_n)$$

is the same as

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x\geq q_{n+1}\}}(q_k)) \times \log(1+\frac{1}{q_k})\right) > B+\log\log\theta(q_{n+1})+\log(1+\frac{1}{q_n})+\log\log\theta(q_n)-\log\log\theta(q_{n+1}).$$

Certainly, if the inequality

$$\log(1 + \frac{1}{q_n}) + \log\log\theta(q_n) \geq \log\log\theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n, then it cannot exist infinitely many prime numbers q_n for which $\mathsf{Dedekind}(q_n)$ do not hold. By contraposition, we know that the Riemann Hypothesis should be true. This is the same as

$$\log\left((1 + \frac{1}{q_n}) \times \log\theta(q_n)\right) \geq \log\log\theta(q_{n+1}).$$

That is equivalent to

$$\log\log\theta(q_n)^{1+\frac{1}{q_n}} \geq \log\log\theta(q_{n+1}).$$

Therefore, the Riemann Hypothesis is true when

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n. □

Lemma 2.5. *The Riemann Hypothesis is true when the inequality*

$$(1 - \frac{0.15}{\log^3 x})^{\frac{1}{x}} \times x^{\frac{1}{x}} \geq 1 + \frac{\log(1 - \frac{0.15}{\log^3 x}) + \log x}{x}$$

is satisfied for all sufficiently large positive numbers x.

Proof. Because of the Lemma 2.4, we know that the Riemann Hypothesis is true when

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n. This is the same as

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_n) + \log(q_{n+1})$$

which is

$$\theta(q_n)^{\frac{1}{q_n}} \geq 1 + \frac{\log(q_{n+1})}{\theta(q_n)}.$$

We use the Proposition 1.2 to show that

$$\theta(q_n)^{\frac{1}{q_n}} > (1 - \frac{0.15}{\log^3 q_n})^{\frac{1}{q_n}} \times q_n^{\frac{1}{q_n}}$$

for a sufficiently large prime number q_n. Under our assumption in this Lemma, we have that

$$(1 - \frac{0.15}{\log^3 q_n})^{\frac{1}{q_n}} \times q_n^{\frac{1}{q_n}} \geq 1 + \frac{\log(1 - \frac{0.15}{\log^3 q_n}) + \log q_n}{q_n}.$$

Using the Propositions 1.1 and 1.3, we only need to show that

$$\begin{aligned}
\frac{\theta(q_n)}{\log q_{n+1}} &\geq n \times (1 - \frac{1}{\log n} + \frac{\log\log n}{4 \times \log^2 n}) \\
&> \eta_{q_n} \times (1 - \frac{1}{\log n} + \frac{\log\log n}{4 \times \log^2 n}) \\
&> \frac{q_n}{\log q_n + \log(1 - \frac{0.15}{\log^3 q_n})}
\end{aligned}$$

for a sufficiently large prime number q_n where

$$\eta_{q_n} = \frac{q_n}{\log q_n} + \frac{q_n}{\log^2 q_n} + \frac{2 \times q_n}{\log^3 q_n} + \frac{5.85 \times q_n}{\log^4 q_n} + \frac{23.85 \times q_n}{\log^5 q_n} + \frac{119.25 \times q_n}{\log^6 q_n} + \frac{715.5 \times q_n}{\log^7 q_n} + \frac{5008.5 \times q_n}{\log^8 q_n}.$$

Certainly, as the prime number q_n increases, the value of $(1 - \frac{1}{\log n} + \frac{\log\log n}{4 \times \log^2 n})$ gets closer to 1 and the inequality $\eta_{q_n} \gg \frac{q_n}{\log q_n + \log(1 - \frac{0.15}{\log^3 q_n})}$ starts to become trivially satisfied. Here, the symbol $\gg$ means "much greater than". However, this implies that

$$\frac{\log(1 - \frac{0.15}{\log^3 q_n}) + \log q_n}{q_n} > \frac{\log(q_{n+1})}{\theta(q_n)}$$

which is equal to

$$1 + \frac{\log(1 - \frac{0.15}{\log^3 q_n}) + \log q_n}{q_n} > 1 + \frac{\log(q_{n+1})}{\theta(q_n)}$$

and finally, the proof is complete. □

Theorem 2.6. *The Riemann Hypothesis is true.*

Proof. From the Proposition 1.7, we have that:

$$\frac{\log(1 - \frac{0.15}{\log^3 x}) + \log x}{x} \geq \log(1 + \frac{\log(1 - \frac{0.15}{\log^3 x}) + \log x}{x})$$

since

$$\frac{\log(1 - \frac{0.15}{\log^3 x}) + \log x}{x} > -1$$

for every sufficiently large positive number x. We know that

$$\begin{aligned}
\frac{\log(1 - \frac{0.15}{\log^3 x}) + \log x}{x} &= \frac{\log\left((1 - \frac{0.15}{\log^3 x}) \times x\right)}{x} \\
&= \log\left((1 - \frac{0.15}{\log^3 x})^{\frac{1}{x}} \times x^{\frac{1}{x}}\right)
\end{aligned}$$

by the properties of the logarithm. This implies that

$$\log((1 - \frac{0.15}{\log^3 x})^{\frac{1}{x}} \times x^{\frac{1}{x}}) \geq \log(1 + \frac{\log(1 - \frac{0.15}{\log^3 x}) + \log x}{x})$$

which is equivalent to

$$(1-\frac{0.15}{\log^3 x})^{\frac{1}{x}} \times x^{\frac{1}{x}} \geq 1+\frac{\log(1-\frac{0.15}{\log^3 x})+\log x}{x}$$

for every sufficiently large positive number x. This final result is a direct consequence of the Lemma 2.5. □

3. Conclusion

The practical uses of the Riemann Hypothesis include many propositions which are known as true under the Riemann Hypothesis, and some that can be shown equivalent to the Riemann Hypothesis. Certainly, the Riemann Hypothesis is closed related to various mathematical topics such as the distribution of prime numbers, the growth of arithmetic functions, the Lindelöf Hypothesis, the large prime gap conjecture, etc. Indeed, a proof of the Riemann Hypothesis could spur considerable advances in many mathematical areas, such as the number theory and pure mathematics in general. We consider that our paper has achieved this goal considered as the Holy Grail of Mathematics by several authors.

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