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Abstract. In this study, we consider the multiple traveling salesmen problem (mTSP) with the min-max objective of minimizing the longest tour length. We begin by reviewing an existing integer programming (IP) formulation of this problem. Then, we present several novel conjunctive normal form (CNF) encodings and an approach based on modifying a maximum satisfiability (MaxSAT) algorithm for the min-max mTSP. The correctness and the space complexity of each encoding are analyzed. In our experiments, we compare the performance of solving the TSP benchmark instances using an existing encoding and our new encodings comparing the results achieved using the proposed extended MaxSAT solver to those achieved using the IP method. The results show that for the same problem, the new encodings significantly reduce the number of generated clauses over the existing CNF encoding. Compared to the IP method, one of the proposals is more effective on relatively large-scale problems, and it also has an obvious advantage over the IP method in solving instances with a small ratio of the number of cities to the number of salesmen.

Keywords: Boolean satisfiability · Min-max optimization · Multiple traveling salesmen problem.

1 Introduction

As one of the classic problems in theoretical computer science, the traveling salesman problem (TSP) has also received much attention in operations research. Moreover, the TSP has been reformulated to address various practical application problems. The multiple TSP (mTSP) is a simple extension of TSP in which more than one salesman is deployed concurrently to visit a set of interconnected cities. All salesmen depart from and return to the same depot. Except for the depot, each of the cities can only be visited exactly once by a single salesman. A broader range of real-life problems can be modeled as the mTSP, including for example, mission planning [5], workload balancing [20], printing press scheduling [11], vehicle routing [12], and ride sharing [16]. Regarding the objective to be

optimized, there are two distinct directions, as follows: one that minimizes the total distance traveled by all the salesmen (*min-sum*), and another that minimizes the distance of the longest tour (*min-max*). Generally speaking, if we consider all tours as a vector, then the min-sum objective is to determine the minimal *Manhattan norm* of the vector, while the min-max objective is to determine its minimal *Chebyshev norm*. However, the min-sum objective conditionally results in highly imbalanced solutions in which one salesman visits all or most of the cities, if no restriction is imposed on the number of cities to be visited by each salesman. Furthermore, as an emphasis on practicality, multi agent cooperation does not aim to reduce costs, but rather to reduce the makespan to serve all the clients. For this reason, we focus in this paper on the min-max mTSP.

1.1 Related Work

From the standpoint of computational complexity theory, the mTSP is strongly $\mathcal{NP}$-hard as the TSP is a special case. In addition, for the min-max optimization problem, some theorems related to its computational complexity have been proved in the following literature. Yu [28] discussed the corresponding min-max version of several classical discrete optimization problems including the minimum spanning tree problem, the resource allocation problem, and the production control problem. The strong $\mathcal{NP}$-hardness of these problems is shown for an unbounded number of scenarios. Ko and Lin [13] presented a number of optimization problems, such as the min-max clique problem, the min-max three-dimensional matching problem, and the min-max circuit problem, and showed that they are complete for the class $\Pi_2^{\mathcal{P}}$, the second level of the polynomial-time hierarchy. Aissi et al. [1] proved that the min-max and min-max regret versions of the assignment problem are strongly $\mathcal{NP}$-hard when the number of scenarios is not bounded by a constant.

Because the min-max mTSP is much more difficult to solve than the min-sum version, only very small-scale instances of the min-max mTSP can be solved optimally within an appropriate time limit. One of the earliest exact algorithms for the min-max mTSP, discussed by França et al. [8], was based on the Tabu search heuristic with the dichotomous and downward search schemes. Despite the important academic and engineering value of the min-max mTSP, the research on it is relatively limited and different heuristic approaches have been developed in the literature. Frederickson et al. [9] proposed some approximation algorithms, which included k-near insert, k-near neighbor, and k-split tour for the min-max mTSP. Somhom et al. [23] and Modares et al. [18] developed a self-organizing neural network approach for the min-max mTSP, which introduced a competition method to decide whether a city should be included in a tour. Soylu [24] presented a general variable neighborhood search algorithm. Necula et al. [19] and Venkatesh and Singh [27] respectively proposed various swarm intelligence algorithms, such as the ant colony, the artificial bee colony, and the invasive weed optimizations for the min-max mTSP. More recently, Vandermeulen et al. [26] formulated combined task assignment and routing problems as the minimum Hamiltonian partition problem, which is equivalent to the min-max mTSP, and developed a heuristic algorithm for solving it.

1.2 Contributions

In Section 3, we propose three CNF encodings for the min-max mTSP in reference to the characteristics of the IP formulation. These three encodings are all to prevent subtours occurring in the solution of the problem; two are based on vertex potential constraints, and the third is based on reachability constraints. For each proposed encoding, we provide a complete proof of its correctness and space complexity. In Section 4, instead of a reduction from the min-max optimization problem to the general weighted partial MaxSAT with an excessive memory requirement, we propose an extended MaxSAT algorithm to solve the encoded min-max mTSP. In Section 5, we compare our proposed CNF encodings and existing naive encoding with an IP approach on the instances of the TSP benchmark. Source code for our experiments is available.[3]

2 Preliminaries

The mTSP is defined on a directed graph $G = (V, E)$, where V is the set of vertices and E is the set of edges. The graph is associated with a distance matrix $D = (d_{ij})$ for each edge $(i, j) \in E$. The matrix D is said to be *symmetric* when $d_{ij} = d_{ji}$, $\forall\,(i, j) \in E$ and *asymmetric* otherwise.

2.1 IP Formulation for the Min-Max mTSP

Owing to the two-dimensional characteristics of edges, the min-sum mTSP is typically formulated using an assignment-based double-index IP formulation, while for the min-max mTSP, a general scheme is to add a third dimension in order to distinguish clearly among the edges assigned to each salesman. Therefore, we let x_{ijk} be a binary variable that is equal to 1 if edge (i, j) is selected in the k-th salesman's tour and 0 otherwise. We also define an integer variable u_i (the *potential* of vertex i) as the number of cities visited on a salesman's path from the depot to city i. Then, the min-max mTSP can be described as follows [19]:

$$\min\ \theta \tag{1}$$

$$\text{s.t.} \sum_{j=2}^{n} x_{1jk} = 1, \qquad k = 1, \ldots, m \tag{2}$$

$$\sum_{i=2}^{n} x_{i1k} = 1, \qquad k = 1, \ldots, m \tag{3}$$

$$\sum_{i=1}^{n} \sum_{k=1}^{m} x_{ijk} = 1, \qquad j = 2, \ldots, n, \quad i \neq j \tag{4}$$

$$\sum_{j=1}^{n} \sum_{k=1}^{m} x_{ijk} = 1, \qquad i = 2, \ldots, n, \quad i \neq j \tag{5}$$

[3] https://drive.google.com/file/d/1Rn-uTPBqoAnUmMMmbAX7lAmyC_ixuetb/view?usp=sharing

$$\sum_{i=1}^{n} x_{ijk} = \sum_{i=1}^{n} x_{jik}, \qquad j = 2, \ldots, n, \quad i \neq j, \quad k = 1, \ldots, m \tag{6}$$

$$u_i - u_j + (n-m) \cdot \sum_{k=1}^{m} x_{ijk} \leq n - m - 1, \qquad 2 \leq i \neq j \leq n \tag{7}$$

$$\sum_{(i,j)\in E} d_{ij} x_{ijk} \leq \theta, \qquad k = 1, \ldots, m \tag{8}$$

$$x_{ijk} \in \{0,1\}, \forall\, (i,j) \in E, \qquad k = 1, \ldots, m$$

where the number of cities (including the depot) is n, the number of salesmen is m with $m \leq n$. Constraint (2) (resp., constraint (3)) guarantees that, for each salesman k, the depot (i.e., vertex 1) is to be departed from (resp., returned to) exactly once. Constraint (4) (resp., constraint (5)) ensures that each non-depot city is to be visited (resp., departed from) exactly once. Constraint (6) enforces that for each non-depot city, the salesman who enters and exits the same city must be consistent. Constraint (7) [25,10] is based on the *subtour elimination constraint* (SEC) proposed by Miller et al. [17], referred to here as the MTZ-based SEC, where the generated formulae and the required vertex potentials are $O(n^2)$. This constraint is used to prevent *subtours*, which are degenerate tours that are formed between non-depot cities and not connected to the depot. In addition, this constraint ensures that each salesman is to visit at least one non-depot city. Here $n - m$ is the maximum number of vertices that can be visited by any salesman. The potential of each vertex indicates the order of the corresponding vertex in the tour. The objective function (1) is to minimize the auxiliary variable θ ($\theta \in \mathbb{R}$) indicating the upper bound of each salesman's tour length, as shown in inequality (8).

Definition 1. *The min-max optimization problem (MMOP) is defined generally as follows:*

$$\min_{k,\epsilon} \theta \quad s.t.\ \mathcal{C} \wedge \bigwedge_{k=1}^{m} \Big(f(k,\epsilon) \leq \theta \Big), \tag{9}$$

where $f(k,\epsilon)$ is the individual cost function for k ($1 \leq k \leq m$), ϵ is a set of other related variables and $\mathcal{C}$ is the set of remaining constraints. Specially, for the min-max mTSP, $f(k,\epsilon) = \sum_{\epsilon\in E} d_\epsilon x_{\epsilon k}$, and $\mathcal{C}$ consists of constraints (2)–(7).

2.2 Maximum Satisfiability

A well-known *Boolean satisfiability problem* (SAT) was the first problem shown to be $\mathcal{NP}$-complete [6]; this problem requires determining whether there exists a truth assignment that satisfies a given Boolean formula.[4] Typically, a Boolean

[4] A truth assignment is a function $\mathcal{A} : X \to \{0,1\}$, where X is a set of Boolean variables and $\mathcal{A}$ is regarded as a conjunction of all elements in X.

formula is expressed in *conjunctive normal form* (CNF), consisting of a conjunction (using the symbol $\wedge$) of one or more clauses. A *clause* is a disjunction (using the symbol $\vee$) of one or more literals, and a *literal* is an occurrence of a Boolean variable or its negation (using the symbol $\neg$).

MaxSAT is an optimal version of SAT [15]. In the *weighted partial* MaxSAT, the problem instance is typically expressed as a set of hard and soft clauses, where each soft clause has a bounded positive numerical *weight* that indicates the cost of falsifying the soft clause. The problem is to find a model that satisfies all the hard clauses and minimizes the total cost (i.e., maximizing the sum of the weights of the satisfied soft clauses). Formally, we denote a MaxSAT formula as $\mathcal{F} = \mathcal{H} \wedge (C_{n+1}, w_1) \wedge \ldots \wedge (C_{n+m}, w_m)$, where $\mathcal{H}$ is the set of hard clauses consisting of $\bigwedge_{i=1}^{n} C_i$ and the remaining clauses (i.e., $\bigwedge_{i=1}^{m} C_{n+i}$) are soft. Solving a MaxSAT instance $\mathcal{F}$ amounts to finding an assignment that satisfies $\mathcal{H}$ and minimizes $\sum_{i=1}^{m}(w_i \neg C_{n+i})$. Technically, we introduce the auxiliary Boolean variable b_i for each soft clause C_{n+i} with the implication of $\neg b_i \rightarrow C_{n+i}$ where $1 \leq i \leq m$, and such a b_i is called the *blocking variable* [14], to ensure that $\mathcal{F}$ can be solved through the resolution of a sequence of SAT instances associated with the *pseudo-Boolean* (PB) constraints encoding [22] as follows:

$$\mathcal{F}_t = \mathcal{H} \wedge \Big(\bigwedge_{i=1}^{m} (C_{n+i} \vee b_i) \Big) \wedge \underset{\text{PB}}{\mathsf{CNF}} \Big(\sum_{i=1}^{m} w_i b_i < t \Big). \tag{10}$$

In Eq. (10), $\mathcal{F}_t$ is a CNF that is satisfiable if and only if $\mathcal{F}$ has an assignment $\mathcal{A}$ whose cost (i.e., $\sum_{i=1}^{m}(w_i \neg C_{n+i})$) is less than t. If the optimal assignment of $\mathcal{F}$ is $\mathcal{A}^*$ and its minimal cost is t^*, then the SAT problem $\mathcal{F}_t$ for $t \geq t^*$ is satisfiable, while the problem for $t < t^*$ is unsatisfiable. For SAT testing a sequence of $\mathcal{F}_t$, t is initialized to $\sum_{i=1}^{m} w_i + 1$, with the next t depending on the current assignment $\mathcal{A}$, obtained from the previous testing. Whenever $\mathcal{F}_t$ is unsatisfiable, t^* is the last tested and satisfiable t. Therefore, to search for the minimal cost for $\mathcal{F}$ is to find the precise location of this transition from satisfiable to unsatisfiable CNF formulae. MaxSAT solvers based on this approach are typically called *satisfiability-based* solvers, the term used in the remainder of this paper.

Definition 2. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of n Boolean variables. The following naive CNF encodings correspond to three special cases of cardinality constraints for X, as follows.*

At most one constraint: $AMO(X) = \bigwedge_{i=1}^{n} \bigwedge_{j=i+1}^{n} (\neg x_i \vee \neg x_j)$.
At least one constraint: $ALO(X) = \bigvee_{i=1}^{n} x_i$.
Exactly-one constraint: $EO(X) = AMO(X) \wedge ALO(X)$.

3 CNF Encodings for the MTZ-Based SEC

With the min-max mTSP in IP formulation, constraints (2)–(5) can be encoded directly into CNF by using the exactly-one constraint. In constraint (6), according to the previous constraints, we know that both $\sum_{i=1}^{n} x_{ijk}$ and $\sum_{i=1}^{n} x_{jik}$ are less

than or equal to one. Therefore, the equation $\sum_{i=1}^{n} x_{ijk} = \sum_{i=1}^{n} x_{jik}$ is equivalent to its logical form $\bigvee_{i=1}^{n} x_{ijk} \leftrightarrow \bigvee_{i=1}^{n} x_{jik}$, which can also be represented simply as CNF. For encoding $\mathcal{C}$ in Eq. (9), the remaining work is to convert the MTZ-based SEC, viz., constraint (7), into a CNF formula.

3.1 Arithmetic Encoding

The most explicit method is to encode the arithmetic formula expressed as constraint (7) directly into CNF. For each vertex potential u_i, $0 \leq u_i \leq n-2$, where $2 \leq i \leq n$ and n is the number of cities. Therefore, in total $(n-1)(n-2)$ Boolean variables are required corresponding to every possible value of all vertex potentials.[5] We denote these Boolean variables as μ_{it}, where $2 \leq i \leq n$ and $1 \leq t \leq n-2$. Then constraint (7) can be rewritten in the following form:

$$\begin{aligned} &\sum_{t=1}^{n-2} \Big(t(\mu_{it} - \mu_{jt}) \Big) + \sum_{k=1}^{m} \Big((n-m) x_{ijk} \Big) \leq n - m - 1 \\ \Leftrightarrow\ &\sum_{t=1}^{n-2} (t\mu_{it}) + \sum_{t=1}^{n-2} (t\neg\mu_{jt}) + \sum_{k=1}^{m} \Big((n-m) x_{ijk} \Big) \leq \frac{n^2 - n}{2} - m \end{aligned} \tag{11}$$

where $2 \leq i \neq j \leq n$. Eq. (11) is a canonical PB constraint that can be encoded into a CNF formula. The arithmetic encoding is based on the idea of transformation from the constraint satisfaction problem (CSP) to SAT.

3.2 Potential Encodings

In addition to encoding the arithmetic expression of constraint (7) directly, we can also achieve the logical conversion according to the specific meaning of the MTZ-based SEC. For any two distinct vertex potentials, the difference between their values must be less than or equal to $n-m-1$. Moreover, $\sum_{k=1}^{m} x_{ijk} = 1$ if and only if vertex j is adjacent to vertex i in the graph. Thus, in this case, we have a pair of vertex potentials for which $u_j - u_i \geq 1$, indicating that vertex j appears after vertex i in the permutation. In brief, the MTZ-based SEC acts to restrict the order of each pair of vertices in its tour if one such pair is the successor of the other one. Obviously, a solution will contradict the exhaustive MTZ-based SECs if it includes any subtour.

Guiding potential We use the properties of vertex potentials described above to introduce a new type of Boolean variables ν_{ijt} to avoid the occurrence of subtours, with $2 \leq i \neq j \leq n$, $(i,j) \in E$, and t $(1 \leq t \leq n-2)$ indicating the potential of the successor vertex j. For example, $\nu_{ijt} = 1$ indicates that a salesman will go directly to city j when departing from city i which is the salesman's

[5] In the case of $u_i = 0$, the corresponding Boolean variable can be omitted.

t-th arrival non-depot city. Therefore, almost $(n-2)(n-1)^2$ additional Boolean variables are required for the following encoding.

$$\bigwedge_{k=1}^{m}\Big(x_{ijk}\rightarrow\bigvee_{t=1}^{n-2}\nu_{ijt}\Big), \qquad 2\le i\neq j\le n \tag{12}$$

$$\bigwedge_{t=1}^{n-2}\Big(\nu_{ijt}\rightarrow\bigvee_{k=1}^{m}x_{ijk}\Big), \qquad 2\le i\neq j\le n \tag{13}$$

$$\bigwedge_{j=2}^{n}\Big(x_{1jk}\rightarrow x_{j1k}\vee\bigvee_{l=2}^{n}\nu_{jl1}\Big), \qquad k=1,\ldots,m,\quad l\neq j \tag{14}$$

$$\bigwedge_{t=1}^{n-2}\left(\nu_{ijt}\rightarrow\bigvee_{k=1}^{m}x_{j1k}\vee\begin{cases}\bigvee_{l=2}^{n}\nu_{jl(t+1)}, & \text{if } t\neq n-2,\\ 0, & \text{otherwise.}\end{cases}\right), \qquad \begin{array}{r}2\le i\neq j\le n,\\ j\neq l\end{array} \tag{15}$$

$$\bigwedge_{j=2}^{n}AMO\Big(\bigcup_{t=1}^{n-2}\bigcup_{i=2}^{n}\nu_{ijt}\Big). \tag{16}$$

Eq. (12) (resp., Eq. (13)) represents the implication from x_{ijk} to $\bigvee_{t=1}^{n-2}\nu_{ijt}$ (resp., from ν_{ijt} to $\bigvee_{k=1}^{m}x_{ijk}$). Eqs. (14) and (15) both guide x_{j1k} or ν_{jlt} in the forward direction of each tour (i.e., the next vertex potential), with the former corresponding to the potential value of 1 and the latter corresponding to the remaining cases. Eq. (16) imposes the restriction that each successor vertex j can correspond only to at most one specific potential.

Theorem 1. *The simultaneous Eqs. (12)–(16) ensure that there are no subtours in the solution of the min-max mTSP.*

Proof. Assume that an assignment $\mathcal{A}$ includes at least one subtour and that the set of edge connections in such subtour is $\{x_{ab\kappa},\ldots,x_{ca\kappa}\}$ (i.e., $\mathcal{A}(x_{ab\kappa}\wedge\ldots\wedge x_{ca\kappa})=1$), where $2\le a\neq b\le n$ and $2\le c\neq a\le n$. This subtour indicates that the salesman κ travels from non-depot city a to b and finally returns to a from c. In accordance with Eq. (12), the projection relation from x_{ijk} to $\bigvee_{t=1}^{n-2}\nu_{ijt}$ is constructed, to ensure that $\bigvee_{t=1}^{n-2}\nu_{ijt}$ can reflect the edge connections of x_{ijk}. Since each route can be regarded as a series of head-to-tail edges with an increasing potential to each tail vertex, we begin by constraining the connection between the first arriving non-depot city and successor city through Eq. (14). The subsequent edge connections are guided according to Eq. (15). An inevitable contradiction will occur on the constraint of the last edge of the subtour. Here we discuss the following two cases when $\mathcal{A}(\nu_{cat})=1$:

- If $t\neq n-2$, then $\nu_{cat}\rightarrow\bigvee_{k=1}^{m}x_{a1k}\bigvee_{l=2}^{n}\nu_{al(t+1)}$.
  - If $\mathcal{A}(\bigvee_{k=1}^{m}x_{a1k})=1$, then the definition of subtour is violated if $\mathcal{A}(x_{a1\kappa})=1$; inconsistencies with constraint (6) arise otherwise.
  - If $\mathcal{A}(\bigvee_{l=2}^{n}\nu_{al(t+1)})=1$, then a contradiction with Eq. (16) arises if $\mathcal{A}(\nu_{ab(t+1)})=1$; otherwise, according to Eq. (13), this conflicts with $\mathcal{A}(\bigvee_{k=1}^{m}x_{alk})=1$ ($l\neq b$) for the same reason as the previous subitem whether $k=\kappa$ or not.

– Otherwise, we have $\nu_{ca(n-2)} \to \bigvee_{k=1}^{m} x_{a1k}$. This might contradict the definition of subtour or another constraint for the same reason as that mentioned in the first subitem of the previous item.

Consequently, solutions of min-max mTSP without any subtour are guaranteed by simultaneous Eqs. (12)–(16). □

Theorem 2. *Eqs. (12)–(16) always produce a polynomial-sized CNF that includes the number of generated clauses of complexity $O(n^5 + mn^2)$, where m is the number of salesmen and n is the number of cities.*

Proof. The number of clauses involved in Eq. (12) is bounded above by $m(n-1)^2$, while that involved in Eq. (13) is bounded by $(n-2)(n-1)^2$. In Eqs. (14) and (15), the number of clauses is bounded, respectively, by $m(n-1)$ and $(n-2)(n-1)^2$. Last, bounded by $(n-1)\binom{(n-2)(n-1)}{2}$ binary clauses are generated in Eq. (16). Therefore, Eqs. (12)–(16) always produce a polynomial-sized CNF that includes the number of generated clauses of complexity $O(n^5+mn^2)$, in which the number of binary clauses is $O(n^5)$. □

Blocking cycle An alternative interpretation of MTZ-based SEC is that constraints prevent cycles with respect to the set of vertices $\{2, \ldots, n\}$. We still use the aforementioned new type of Boolean variables ν_{ijt}, but we replace the vertex potential guidance and some related constraints including Eqs. (13)–(16) with the following constraints to block cycles.

(12)

$$\bigwedge_{i=2}^{n} \Big(\bigvee_{j=2}^{n} \nu_{ijt} \to \bigwedge_{\tau=t}^{n-2} \bigwedge_{l=2}^{n} \neg \nu_{li\tau} \Big) \qquad t = 1, \ldots, n-2, \quad j \neq i, \quad l \neq i \tag{17}$$

Eq. (17) indicates that, for potentials τ greater than or equal to the current potential t, the head of any edge connection (i.e., departure vertex) i can not be the tail of any edge connection (i.e., successor vertex).

Theorem 3. *The simultaneous Eqs. (12) and (17) guarantee that there are no subtours in the solution of the min-max mTSP.*

Proof. Assume that an assignment $\mathcal{A}$ includes at least one subtour and consider that the set of edge connections in such a subtour is $S = \{x_{ab\kappa}, \ldots, x_{ca\kappa}\}$ (i.e., $\mathcal{A}(x_{ab\kappa} \wedge \ldots \wedge x_{ca\kappa}) = 1$), where $2 \leq a \neq b \leq n$ and $2 \leq c \neq a \leq n$. This subtour indicates that the salesman κ starts from non-depot city a to b and finally returns to a from c. According to Eq. (12), $\mathcal{A}(\bigvee_{t=1}^{n-2} \nu_{abt} \wedge \ldots \wedge \bigvee_{t=1}^{n-2} \nu_{cat}) = 1$. Now focus on $\mathcal{A}(\nu_{abt_1} \wedge \ldots \wedge \nu_{cat_{|S|}}) = 1$, where t_1 (resp., $t_{|S|}$) is the minimal t with respect to $\mathcal{A}(\bigvee_{t=1}^{n-2} \nu_{abt}) = 1$ (resp., $\mathcal{A}(\bigvee_{t=1}^{n-2} \nu_{cat}) = 1$) and $1 \leq t_1 < \ldots < t_{|S|} \leq n-2$ corresponding to the order of the assumed subtour's edge connections. Further according to Eq. (17), $\mathcal{A}(\bigwedge_{\tau=t_1}^{n-2} \bigwedge_{l=2}^{n} \neg \nu_{la\tau}) = 1$. From $\nu_{cat_{|S|}} \in \bigcup_{\tau=t_1}^{n-2} \bigcup_{l=2}^{n} \nu_{la\tau}$, it follows that $\mathcal{A}(\neg \nu_{cat_{|S|}}) = 1$, which conflicts with the previous derivation. Therefore, solutions of min-max mTSP without any subtour can be ensured by Eqs. (12) and (17). □

Theorem 4. *Eqs. (12) and (17) always produce a polynomial-sized CNF that includes the number of generated clauses of complexity $O(n^5 + mn^2)$, where m is the number of salesmen and n is the number of cities.*

Proof. The number of clauses required in Eq. (12) is bounded above by $m(n-1)^2$. In Eq. (17), the number of clauses is bounded by $\frac{(n-2)(n-1)^4}{2}$ and all these clauses are binary clauses. Therefore, Eqs. (12) and (17) also produce a polynomial-sized CNF that includes the number of generated clauses of complexity $O(n^5 + mn^2)$, in which the number of binary clauses is of complexity $O(n^5)$. □

3.3 Relative Encoding

To prevent cycles other than the main tour for each salesman, we also propose an encoding method based on reachability constraints. In this encoding, we can compact the variables of edge connections further from triple-index to double-index. In contrast to the previous x_{ijk} shown in Fig. 1 (a), we define a new type of Boolean variables l_{ij} depicted in Fig. 1 (b). For the axes of i and j, instead of using the first index (highlighted red in Fig. 1 (a)) to represent the depot city in x_{ijk}, we use the first m indices (highlighted red in Fig. 1 (b)) to represent m duplications of the depot city sequentially for every salesmen in l_{ij}.

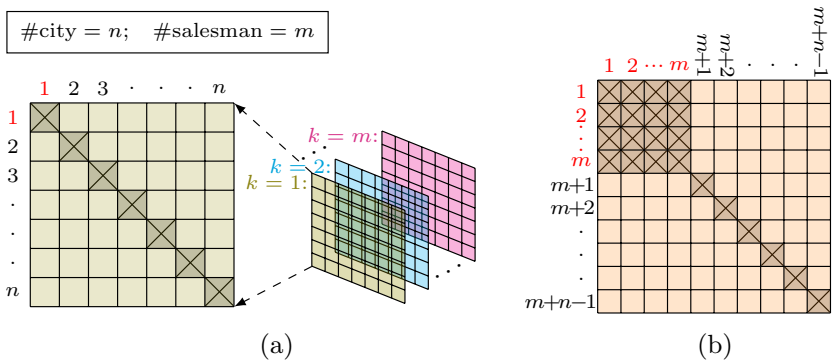

Fig. 1. Boolean variables of edge connections: (a) Each cell of a matrix indicates the variable x_{ijk} with the specific i (index of rows), j (index of columns), and k (index of salesmen), where for both rows and columns, index 1 corresponds to the depot city. (b) Each cell of the matrix indicates the variable l_{ij} with the specific i (index of rows) and j (index of columns), where for both rows and columns, indices from 1 to m, respectively, correspond to the depot city for each salesman k. The cells marked by ⊠ indicate that the corresponding variables can be omitted.

In addition, we also introduce another new type of Boolean variable r_{ij} which indicates whether vertex j can be reached via vertex i, where $1 \leq i \leq n$, $m+1 \leq j \leq m+n-1$, and $i \neq j$. In other words, $l_{ij} = 1$ if and only if vertex j appears immediately after vertex i in any salesman's tour; while $r_{ij} = 1$ if

and only if vertex i appears before vertex j in any salesman's tour. This idea, based on the relative positions of vertices in the permutation, was first proposed in Prestwich [21] and was devoted to a CNF encoding of the *Hamiltonian path problem*. Therefore, $O(m^2+n^2+mn)$ Boolean variables are required for encoding $\mathcal{C}$ in Eq. (9), as follows.

$$\bigwedge_{i=1}^{m} EO\Big(\bigcup_{j=m+1}^{m+n-1} l_{ij}\Big), \tag{18}$$

$$\bigwedge_{j=1}^{m} EO\Big(\bigcup_{i=m+1}^{m+n-1} l_{ij}\Big), \tag{19}$$

$$\bigwedge_{i=m+1}^{m+n-1} EO\Big(\bigcup_{j=1}^{m+n-1} l_{ij}\Big), \qquad j \neq i \tag{20}$$

$$\bigwedge_{j=m+1}^{m+n-1} EO\Big(\bigcup_{i=1}^{m+n-1} l_{ij}\Big), \qquad i \neq j \tag{21}$$

$$\bigwedge_{i=1}^{m+n-1} \bigwedge_{j=m+1}^{m+n-1} (l_{ij} \rightarrow r_{ij}), \qquad j \neq i \tag{22}$$

$$\bigwedge_{i=1}^{m+n-1} \bigwedge_{j=m+1}^{m+n-1} \bigwedge_{k=m+1}^{m+n-1} (r_{ij} \wedge r_{jk} \rightarrow r_{ik}), \qquad \text{condition set} \tag{23}$$

$$\bigwedge_{i=m+1}^{m+n-1} \bigwedge_{j=i+1}^{m+n-1} (\neg r_{ij} \vee \neg r_{ji}). \tag{24}$$

Eq. (18) (resp., Eq. (19)) indicates that for each salesman, there exists exactly one departure from (resp., return to) the depot city to (resp., from) another city. Eq. (20) (resp., Eq. (21)) specifies that each non-depot city can be departed from (resp., visited) exactly once. Eq. (22) shows the implication from l_{ij} to r_{ij}. The transitive law of reachability variables is given by Eq. (23), where the condition set includes that $k \neq j \neq i$ and among the three indices i, j, and k, at most one of them is less than or equal to m (i.e., indicating the depot city). Last but not least, Eq. (24) is an acyclic constraint.

Because this encoding does not include an equation that corresponds to constraint (6), the consistency of a salesman who enters and exits the same city can be restricted only by reachability constraints. However, the Boolean variables r_{ij} range over $m+1 \leq j \leq m+n-1$. Consequently, the last edge connection of each salesman's tour that returns to the depot might not be able to maintain this consistency. For example, we might obtain a solution in which $\mathcal{A}(l_{\kappa a} \wedge l_{ab} \wedge \ldots \wedge l_{c\kappa'}) = 1$, indicating that the tour originates from salesman κ's depot to city a, then goes to the next cities $b, \ldots, c$, and finally returns to the depot of salesman κ'. Here $l_{c\kappa'}$ is the last edge connection in this tour. Such inconsistency does not affect the optimization results, since they have an identical distance cost (i.e., $d_{c\kappa} = d_{c\kappa'}$).

Theorem 5. *The simultaneous Eqs. (18)–(24) guarantee that except for the edge connection that returns the depot for each salesman, the edge connections are a partial solution of the min-max mTSP without any subtour.*

Proof. Assume that an assignment $\mathcal{A}$ includes at least one subtour, consider that the set of edge connections in such subtour is $\{l_{ab}, \ldots, l_{ca}\}$, and suppose that they are both dominated by the salesman κ (i.e., $\mathcal{A}(l_{ab} \wedge \ldots \wedge l_{ca} \wedge r_{\kappa a} \wedge \ldots \wedge r_{\kappa c}) = 1$), where $m+1 \leq a \neq b \leq m+n-1$ and $m+1 \leq c \neq a \leq m+n-1$. According to Eq. (22), we have $\mathcal{A}(r_{ab} \wedge \cdots \wedge r_{ca}) = 1$. Then according to Eq. (23), we can obtain $\mathcal{A}(r_{ac}) = 1$, and this contradicts Eq. (24). □

Lemma 1. *Eqs. (22)–(24) ensure that all reachability variables r_{ij} can be excluded from decision variables in the entire inference process of solving the min-max mTSP.*

Proof. In Theorem 5, every reachability literal r_{ij} or $\neg r_{ij}$ involved in its proof are evaluated to 1 by unit propagations according to Eqs. (22)–(24). Therefore, we can consider all r_{ij} as the support variables used to restrict the solution without any subtour and declare them to be non-decision variables. □

Theorem 6. *Eqs. (18)–(24) always produce a polynomial-sized CNF that includes a number of generated clauses of complexity $O(n^3 + mn^2)$, where m is the number of salesmen and n is the number of cities.*

4 An Extended MaxSAT Algorithm

To solve the MMOP shown in Eq. (9), we began by transforming it successfully into general MaxSAT. However, the reduction method suffers from execution slow down or even memory-out caused by the huge size of the encoded formula, which introduces too many clauses with auxiliary variables owing to the totalizer (TO) encoding [2,3]. This section proposes another approach to solving the MMOP. Unlike the reduction method, this approach describes the MMOP as a problem that slightly extends the DIMACS format of the weighted partial MaxSAT problem, and solves it using a corresponding extended algorithm.

4.1 Grouped Soft Clauses

As has been mentioned in Definition 1, for the min-max mTSP, $f(k, \epsilon)$ $(1 \leq k \leq m)$ can be expressed as a linear adder, where $f(k, \epsilon) = \sum_{i=1}^{n} \sum_{j=1}^{n} (d_{ij} \cdot x_{ijk})$ for the potential encodings and $f(k, \epsilon) = \sum_{i=1}^{m+n-1} \sum_{j=1}^{m} (d_{ij} \cdot r_{ki} \cdot l_{ij}) + \sum_{i=1}^{m+n-1} \sum_{j=m+1}^{m+n-1} (d_{ij} \cdot r_{kj} \cdot l_{ij})$ for the relative encoding. Note that, due to the difference between the directed graphs corresponding to the potential encodings and the relative encoding (see Fig. 1), the distance matrix $D = (d_{ij})$ for the relative encoding is an extension of that for the potential encodings. However, the standard MaxSAT formula (i.e., the DIMACS format) simply divides all constraints into two parts: hard clauses and soft clauses, with no further subdivisions

for soft clauses. To translate a MMOP into an extended MaxSAT problem, we formulate the modified MaxSAT formulae respectively for the potential encodings and for the relative encoding as follows:

$$\begin{aligned}
\mathcal{F}'_{(\text{potential})} &= \mathcal{H} \wedge \bigwedge_{k=1}^{m} \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} (\neg x_{ijk}, d_{ij}, k), \\
\mathcal{F}'_{(\text{relative})} &= \mathcal{H} \wedge \bigwedge_{k=1}^{m} \bigwedge_{i=1}^{m+n-1} \Big(\bigwedge_{j=1}^{m} (\neg r_{ki} \vee \neg l_{ij}, d_{ij}, k) \bigwedge_{j=m+1}^{m+n-1} (\neg r_{kj} \vee \neg l_{ij}, d_{ij}, k) \Big).
\end{aligned} \tag{25}$$

Each grouped soft clause can be regarded as a triple $(\neg \Gamma, d_{ij}, k)$ indicating that the constraint $\neg \Gamma$ corresponds to the weight d_{ij} and is labeled as the k-th group of soft clauses. This modification allows us to distinguish between extended soft clauses in accordance with their respective groups.

4.2 Multiple PB Constraints for $f(k, \epsilon) \leq \theta$

To solve the modified MaxSAT formulae $\mathcal{F}'$ in Eq. (25), we encode it into a series of SAT instances referring to Eq. (10) by simultaneously using multiple PB constraints encoding as follows:

$$\begin{aligned}
\mathcal{F}'_{t(\text{potential})} &= \mathcal{H} \wedge \bigwedge_{k=1}^{m} \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} (\neg x_{ijk} \vee b_{ijk}) \wedge \bigwedge_{k=1}^{m} \underset{\text{PB}}{\mathsf{CNF}} \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} (d_{ij} \cdot b_{ijk}) < t \Big) \\
&= \mathcal{H} \wedge \bigwedge_{k=1}^{m} \underset{\text{PB}}{\mathsf{CNF}} \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} (d_{ij} \cdot x_{ijk}) < t \Big), \\
\mathcal{F}'_{t(\text{relative})} &= \mathcal{H} \wedge \bigwedge_{k=1}^{m} \bigwedge_{i=1}^{m+n-1} \Big(\bigwedge_{j=1}^{m} (\neg r_{ki} \vee \neg l_{ij} \vee b_{ijk}) \bigwedge_{j=m+1}^{m+n-1} (\neg r_{kj} \vee \neg l_{ij} \vee b_{ijk}) \Big) \wedge \\
&\quad \bigwedge_{k=1}^{m} \underset{\text{PB}}{\mathsf{CNF}} \Big(\sum_{i=1}^{m+n-1} \sum_{j=1}^{m+n-1} (d_{ij} \cdot b_{ijk}) < t \Big) \\
&= \mathcal{H} \wedge \bigwedge_{k=1}^{m} \underset{\text{PB}}{\mathsf{CNF}} \left(\sum_{i=1}^{m+n-1} \Big(\sum_{j=1}^{m} (d_{ij} \cdot r_{ki} \cdot l_{ij}) + \sum_{j=m+1}^{m+n-1} (d_{ij} \cdot r_{kj} \cdot l_{ij}) \Big) < t \right).
\end{aligned} \tag{26}$$

Theorem 7. *Whether for the potential encodings or the relative encoding, in Eq. (26), $\mathcal{F}'_t$ is a CNF that is satisfiable if and only if $\mathcal{F}'$ (in Eq. (25)) has a valid assignment whose costs for every group (i.e., $\forall k$ $f(k, \epsilon)$) are less than t at the same time.*

Proof. Let's consider that for the potential encodings, the auxiliary variables $\Gamma \rightarrow x_{ijk}$; while for the relative encoding, $\Gamma \rightarrow (r_{ki} \wedge l_{ij})$ if $1 \leq j \leq m$ and $\Gamma \rightarrow (r_{kj} \wedge l_{ij})$ otherwise.

Assume that there exists an assignment $\mathcal{A}$ satisfying $\mathcal{F}'_t$. For any variable b_{ijk} included in soft clauses, if $\mathcal{A}(b_{ijk}) = 1$, then for $\mathcal{F}'$, there exists an assignment $\mathcal{A}'$ consistent with $\mathcal{A}$ such that $\mathcal{A}'(\Gamma) \in \{0, 1\}$. In this case, whether $\mathcal{A}'(\Gamma)$ is 0 or 1, $\sum_i \sum_j (d_{ij} \cdot \Gamma) < t$. If $\mathcal{A}(b_{ijk}) = 0$, then for $\mathcal{F}'$, there exists an assignment $\mathcal{A}'$ consistent with $\mathcal{A}$ such that $\mathcal{A}'(\Gamma) = 0$.

Assume that there exists an assignment $\mathcal{A}$ satisfying $\mathcal{F}'$. For any variable Γ included in soft clauses, if $\mathcal{A}(\Gamma) = 1$, then for $\mathcal{F}'_t$, there exists an assignment $\mathcal{A}'$ consistent with $\mathcal{A}$ such that $\mathcal{A}'(\Gamma) = 1$, implying $\mathcal{A}'(b_{ijk}) = 1$. If $\mathcal{A}(\Gamma) = 0$, then for $\mathcal{F}'_t$, there exists an assignment $\mathcal{A}'$ consistent with $\mathcal{A}$ such that $\mathcal{A}'(b_{ijk}) \in \{0, 1\}$, subsuming $\mathcal{A}'(b_{ijk}) = 0$, with the result that $\sum_i \sum_j (d_{ij} \cdot b_{ijk}) < t$. □

Therefore, according to Theorem 7 we can know definitively that the solution of the extended MaxSAT formula $\mathcal{F}'$ is equivalent to the solution of the original MMOP in Eq. (9).

5 Implementation and Evaluation

We evaluated the existing arithmetic encoding, the proposed encodings, and the IP method experimentally on an Intel Core i7-6850K, 3.6 GHz processor with 32 GB RAM running Ubuntu 18.04. The arithmetic encoding is implemented by using the TO encoding of PB constraints. The proposed encodings are two potential encodings based on guiding potential and blocking cycle, respectively, and a relative encoding. In the experiments, we called these comparison methods arithmetic, guide, acyclic, and ip, respectively, for short.

Since there is no open benchmark for the min-max mTSP, we selected several benchmark instances of different sizes from TSPLIB.[6] For each selected instance, we specified different numbers of salesmen and generated their corresponding problem instances through implemented encodings and IP formulations. We used a state-of-the-art IP optimizer–CPLEX [7], version 12.7.1, and an extended MaxSAT solver by modifying QMAXSAT [29] to solve all generated problem instances. For further performance comparison, we also introduced a heuristic to the extended MaxSAT solver. The heuristic is to average naively the number of cities assigned to each salesman, and it can be thought of as pre-processing, which is only enabled when solving the initial solution. Each generated problem instance will be solved by the corresponding solver within a one-hour time limit.

We selected small-scale instances and compared their runtimes using each method, as shown in Table 1. In contrast, for those instances where the optimal solution cannot be obtained within 3,600 CPU seconds, we compared their final optimized values under various methods, as shown in Table 2. For Tables 1 and 2, the notation "ins$n$_m" in the first column indicates that the number of cites is n and the number of salesmen is m, where "ins$n$" corresponds to the name of the original benchmark instance. Each cell corresponds to its instance (its row) and its encoding/formulation (its column). Except for the cells in the last column, each of cell contains three values, unless an exception occurs such as a time-out

[6] http://comopt.ifi.uni-heidelberg.de/software/TSPLIB95/tsp/

Table 1. Comparison of runtimes using each method

Instance	arithmetic			guide			acyclic			relative			ip
burma14_2	$2.5E^6$	353 s	(342 s)	$1.5E^5$	355 s	(**220 s**)	$1.6E^5$	334 s	(249 s)	$5.4E^3$	t.o.[1]	(t.o.[1])	18 s
burma14_3	$2.8E^6$	46 s	(44 s)	$1.6E^5$	**13 s**	(**13 s**)	$1.7E^5$	22 s	(25 s)	$6.4E^3$	t.o.[1]	(t.o.[2])	5 s
ulysses16_2	$6.0E^6$	1509 s	(1226 s)	$3.1E^5$	270 s	(245 s)	$3.2E^5$	131 s	(**117 s**)	$8.1E^3$	t.o.[2]	(t.o.[2])	910 s
ulysses16_3	$6.5E^6$	1866 s	(1853 s)	$3.3E^5$	396 s	(392 s)	$3.4E^5$	**129 s**	(136 s)	$9.4E^3$	t.o.[1]	(t.o.[1])	t.o.[1]

(1) t.o.[1]: time-out but has reached the optimum value already; (2) t.o.[2]: time-out and has not reached the optimum value yet.

Table 2. Comparison of optimized values using various methods

Instance	arithmetic			guide			acyclic			relative			ip
gr24_2	$7.6E^7$	1098	(1008)	$2.8E^6$	862	(**824**)	$2.9E^6$	840	(831)	$2.7E^4$	2609	(1797)	770
gr24_3	$8.1E^7$	909	(861)	$2.8E^6$	767	(787)	$2.9E^6$	766	(**749**)	$3.0E^4$	814	(827)	648
gr24_4	$8.6E^7$	757	(803)	$2.9E^6$	**660**	(668)	$3.0E^6$	674	(676)	$3.3E^4$	750	(744)	629
gr24_5	$9.1E^7$	697	(670)	$3.0E^6$	642	(640)	$3.1E^6$	632	(**629**)	$3.6E^4$	676	(673)	594
bays29_2	$2.5E^8$	n.a.y.	(3186*)	$7.6E^6$	1495	(1453)	$7.8E^6$	1597	(**1436**)	$4.8E^4$	2393	(2903)	1122
bays29_4	$2.7E^8$	1543	(1614*)	$7.8E^6$	1018	(1041)	$8.1E^6$	1008	(**975**)	$5.7E^4$	1671	(1174)	762
bays29_6	$3.0E^8$	1114	(1116)	$8.3E^6$	807	(821)	$8.5E^6$	795	(**793**)	$6.6E^4$	948	(887)	705
bays29_8	$3.2E^8$	958	(845)	$8.9E^6$	731	(714)	$9.1E^6$	**697**	(702)	$7.6E^4$	779	(766)	684
dantzig42_2		-		$5.3E^7$	2247*	(**549***)	$5.4E^7$	1702*	(**549***)	$1.5E^5$	856	(624*)	482
dantzig42_4		-		$5.4E^7$	1438	(**487***)	$5.5E^7$	614	(**487***)	$1.7E^5$	568	(488*)	473
dantzig42_6		-		$5.5E^7$	769	(**429***)	$5.6E^7$	479	(**429***)	$1.8E^5$	490	(471)	395
dantzig42_8		-		$5.7E^7$	506	(**391***)	$5.8E^7$	435	(**391***)	$2.0E^5$	427	(440)	373
berlin52_3		-		$1.6E^8$	n.a.y.	(8706*)	$1.6E^8$	n.a.y.	(8706*)	$2.9E^5$	**6719**	(7974)	3754
berlin52_6		-		$1.6E^8$	6879	(5379)	$1.7E^8$	7359	(5418*)	$3.4E^5$	4758	(**4721**)	3265
berlin52_9		-		$1.7E^8$	18864*	(3949)	$1.7E^8$	**3842**	(3956*)	$3.8E^5$	3915	(3967)	2628
berlin52_12		-		$1.8E^8$	4795	(3828)	$1.8E^8$	3812	(3184)	$4.3E^5$	**2936**	(3242)	2668
eil76_4		-			-			-		$9.3E^5$	**438**	(514)	333
eil76_8		-			-			-		$1.1E^6$	**345**	(348)	204
eil76_12		-			-			-		$1.2E^6$	**255**	(290)	210
eil76_16		-			-			-		$1.3E^6$	**226**	(229)	245
rat99_5		-			-			-		$2.1E^6$	1864	(**1182***)	1004
rat99_10		-			-			-		$2.3E^6$	1005	(**568***)	912
rat99_15		-			-			-		$2.6E^6$	862	(**621***)	773
rat99_20		-			-			-		$2.9E^6$	826	(**527***)	n.a.y.
bier127_6		-			-			-		$4.4E^6$	139764	(**128779***)	63801
bier127_12		-			-			-		$4.9E^6$	98107	(**71474***)	n.a.y.
bier127_18		-			-			-		$5.5E^6$	75242	(**50742***)	n.a.y.
bier127_24		-			-			-		$6.0E^6$	70310	(**48294***)	n.a.y.
pr152_8		-			-			-		$7.7E^6$	953386*	(**106741***)	105874*
pr152_16		-			-			-		$8.7E^6$	152925	(**69218***)	n.a.y.
pr152_24		-			-			-		$9.8E^6$	129144	(**50343***)	n.a.y.
pr152_32		-			-			-		$1.1E^7$	211209*	(**52308***)	n.a.y.
tsp225_11		-			-			-		$2.5E^7$	1068331	(**314578***)	n.a.y.
tsp225_22/ _33/_44		-			-			-		$2.8E^7$ /$3.1E^7$/$3.5E^7$	m.o.		n.a.y.

(1) m.o.: memory-out; (2) n.a.y.: no answer yet; (3) `value`*: this `value` is the initial solution and it is not updated until the end of the limit time.

or memory-out issue. The value on the left is the number of generated clauses, the middle value is performed by the extended MaxSAT solver, and the value on the right surrounded by round brackets is performed by the extended MaxSAT solver with heuristic pre-processing. For the last column, each cell shows the value performed by CPLEX. In Table 2, the cell marked by - indicates that the

Table 3. The expansion comparison of optimized vaules for instance eil76

eil76	_17	_18	_19	_20	_21	_22	_23	_24	_25	_26	_27	_28	_29	_30	_31	_32	_33	_34	_35	_36	_37	_38
relative	221	214	206	184	201	201	186	196	187	182	168	184	170	173	156	151	164	138	152	168	171	176
ip	211	247	190	215	335	175	269	246	155	225	179	265	310	223	356	170	169	210	147	141	162	179

size of its instance is too large, resulting in a file generated by its encoding being much larger than 10 GB. Thus, it was excluded from our experiment. In our experimental results data, we marked the best performance of all CNF encodings in bold, and if it is better than the ip, it will be highlighted in red.

As the results from Tables 1 and 2 show, the number of clauses generated by various encoding methods is consistent with the theorems we have provided. For the same instance, the number of clauses generated by arithmetic is much larger than that of guide or acyclic, while those of guide and acyclic are approximately the same but also much larger than those of relative. For relatively small-scale instances, both in runtime and optimized value comparisons, acyclic is better than the other encoding methods, while in comparisons of optimized value, there is a disadvantage with ip. However, we found that as the scale of the problem became larger, the performance of relative gradually became better, even surpassing that of ip especially on larger instances where other encoding methods are difficult to handle. For larger instances, the effect of heuristic pre-processing becomes more apparent; even the obtained initial solution is not updated until the end, while ip has no initial solution. Such instances require a runtime far exceeding the provided limit time for better evaluation. The results also show that heuristics are essential for solving the min-max mTSP problem. In addition, we noticed that the smaller ratio of the number of cities to the number of salesmen, the smaller the gap between the SAT-based methods and ip. Consequently, we conducted an expanded experiment to increase the number of salesmen of benchmark instance eil76, shown in Table 3. This experiment compared the performance of the extended MaxSAT solver without heuristic with ip. We can see from Table 3 that, for the instances with a small ratio of the number of cities to the number of salesmen, relative outperformed ip in the overall comparison of optimized values.

6 Conclusion

In this paper, we proposed three CNF encodings for the min-max mTSP. These three encodings are all intended to prevent subtours occurring in the solution of the problem; two of them are based on the vertex potentials, and the third is based on the reachabilities. We also proposed an extended MaxSAT algorithm with an optional heuristic pre-processing to solve the encoded min-max mTSP. In terms of the space complexity of the generated problem, our new encodings are significantly improved over the existing encoding. Furthermore, although the proposed approaches are not as effective as the IP method in the performance of small-scale problems, one of them outperforms the IP method for the instances for which the ratio of the number of cities to the number of salesmen is small.

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